Lance Burger

TEACHING PROPORTIONAL REASONING THROUGH FAMILIAR BIOLOGY

Exploring relationships between size and heat loss in dogs teaches students about the ratio of surface area to volume.

Constructivism has constituted a predominant philosophical trend in mathematics education, but its implementation in the classroom has proven to be "far more difficult than the reform community acknowledges" (Windschitl 2002, p. 131). This difficulty may be due in part to administrative cultures that might insist on more procedural types of instruction aimed at testing, in part to the lack of an accessible synthesis of constructivist theory useful for the everyday teacher. Cobb (1994) defines constructivism as the "generally accepted view that students actively construct their mathematical ways of knowing as they strive to be effective by restoring coherence to the worlds of their personal experience" (p. 13).

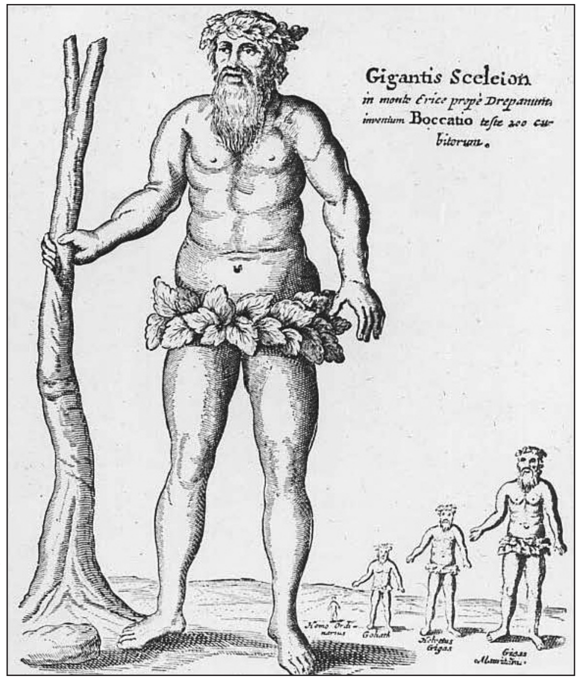
The psychologist Jean Piaget built his theory of learning largely on the idea that "knowledge develops as a solution to a problem" (1970). Piaget's meaning of *problem*, however, is not simply any problem but a problem as seen from a learner's personal perspective—or, to echo Cobb, the world of the learner's personal experience. In constructivist terms, encountering a personally meaningful and significant problem throws the learner into a state of "cognitive disequilibrium," and she must modify her existing knowledge structures (assimilation) to "restore coherence" to solve "her" problem (accommodation).

Inspired by Piaget, Harel (2000) simplified the technical jargon of constructivism into a simple necessity principle: "For students to learn, they must see an (intellectual, as opposed to social or economic) need for what they are intended to be taught" (p. 185). Often, mathematics activities are not difficult, engaging, or meaningful enough to evoke in students a feeling of needing to solve a problem. When lessons are designed with the necessity principle in mind, students should seldom ask, "Why do we need to know this?" Students who see the necessity in the mathematics they learn are more apt to become engaged in and "own" their mathematical problems—a first step for genuinely constructing mathematical knowledge.

Here I describe a lesson implemented using the necessity principle to plan engaging content, discussions, and activities meant to promote meaningful student growth in mathematical understanding.

THERE MIGHT BE GIANTS?

Do organisms' sizes have limits? Could Boccatio's 300-foot giant described by Rose (2000, p. 140) ever have existed (see **fig. 1**)? The teacher begins this lesson with speculation on the



GODWIN (1979), P. 91

Fig. 1 Could giants such as those depicted here ever exist?

possible existence of giants, in addition to presenting work from *Dialogues Concerning Two New Sciences,* in which Galileo comments, "Who does not know that a horse falling from a height of three or four cubits will break his bones, while a dog falling from the same height or a cat from a height of eight or ten cubits will suffer no injury?" (Hawking 2002, p. 3).

Galileo goes on to say that if a man's height be increased inordinately, he will fall and be crushed under his own weight. Although such insights may offer interesting structural observations about giants, students can also explore the improbability of giants through the ratio of surface area to volume in the context of heat exchange in biology. From the very small to the very large, biology has made many adaptations related to the ratio of surface area to volume (Schmidt-Nielson 1984). By applying the necessity principle in lesson design, teachers can integrate familiar real-world knowledge with proportional reasoning, giving students enriching opportunities to investigate the possibility of giants from the mathematical perspective of the surface-area-to-volume ratio.

OF DOGS HOT AND COLD: AN INTERESTING QUESTION

Although this lesson begins with several engaging questions about giants, we continue by talking about dogs of various sizes (because most students are familiar with dogs) to establish a context for an interesting question. After students finish sharing what kinds of dogs they have, the teacher presents several photographs that show large and small dogs

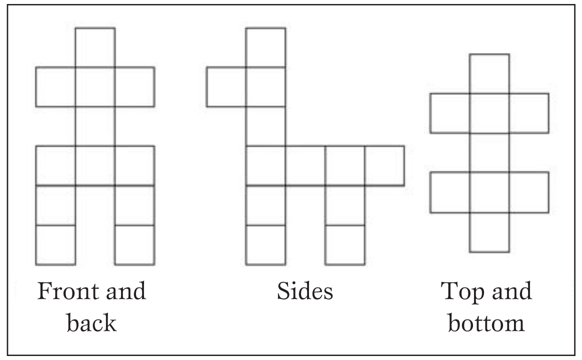


Fig. 2 Net diagrams provide the blueprint for "puppy" construction.

in various environments. Some photographs show large dogs indoors and panting, whereas puppies of the same breed appear indoors and nonpanting, even when bundled up with cozy blankets. Other photographs depict large nonpanting dogs in the snow, whereas dogs of the same breed and size are panting while in warm surroundings. To avoid alienating cat owners, the teacher also shows several photographs depicting panting big cats, such as tigers and lions in a hot-looking African savannah, in contrast to small, nonpanting house cats and kittens in comfortable indoor environments.

The students discuss their observations about the photographs and how they relate to their own experiences with their pets. From these discussions, a fundamental class question usually emerges in the form of the Cooling Dog problem: Why do large dogs appear more overheated and pant more than small dogs?

THE DOUBLE-DOG DARE

For the next part of the lesson, students work with multilink cubes to build models of small and large dogs for hands-on reasoning. First, students construct a "puppy" by using several two-dimensional net diagrams as a blueprint (see **fig. 2**). Because the puppy models do not require many cubes, students construct these models individually. In working from the net diagrams, students usually make several types of errors. For example, the puppies in **figure 3** are too literally based on front, back, and side views. **Figure 4** similarly shows a puppy that matches front and side views but does not fit top and bottom views. Working from the net diagrams helps students develop preliminary concrete visualization skills to distinguish the characteristics of surface area and volume needed for later discussions, proportional reasoning, and computations.

After students complete the puppy models individually, they work in groups to construct a larger version, doubled in length, width, and height. One point regarding the use of manipulatives and the necessity principle: Necessity drives the "need" for groups. This lesson warrants using groups

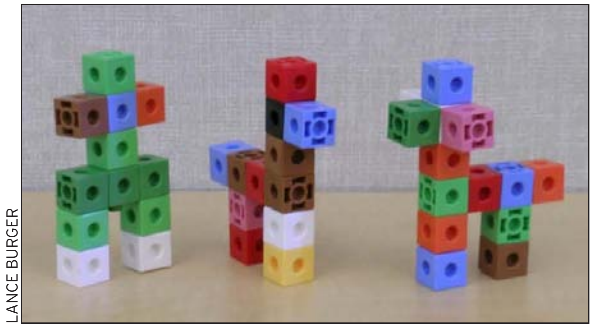


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Fig. 3 Each puppy shown fails to fulfill some of the requirements.

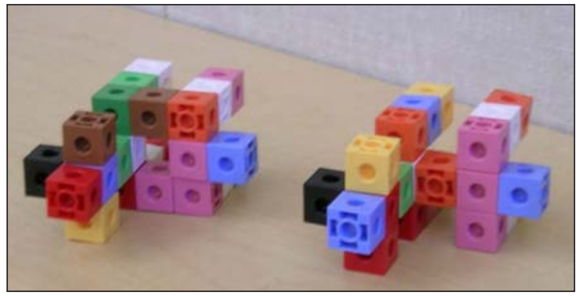

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Fig. 4 Smaller building blocks are assembled to make the larger construction.

because of the significantly greater number of blocks involved and the questions students often have about doubling the size of the puppy. (I avoid designing lessons with group work unless a meaningful need exists for more hands and minds to accomplish specific lesson goals.)

Although the instructions to double the puppies in length, width, and height are explicit, various issues often arise during the activity. Some students double only the height; others double two dimensions but not the third. Usually several groups isolate one part of the puppy, such as the tail or nose, and determine that it consists of one cube of eight blocks. To groups experiencing difficulty, the teacher may have to enlarge a photograph of a student's puppy with a document camera to show that the models of small and large dogs appear qualitatively similar. When the students have completed the large and small dogs, they are ready to bridge mathematical ideas to their models in the context of the overall lesson problem (see **fig. 5**).

RELATING VOCABULARY TO MODELS

To relate vocabulary to models with minimal teacher involvement, students remain in their groups and brainstorm about familiar mathematics concepts that may be associated with their models in the context of the Cooling Dog problem. Terms such as *mass, density, weight, length, width, height, surface area,* and *volume* commonly arise in group discussions. Also at this time, it is helpful for students to review quickly the meanings of their brain-

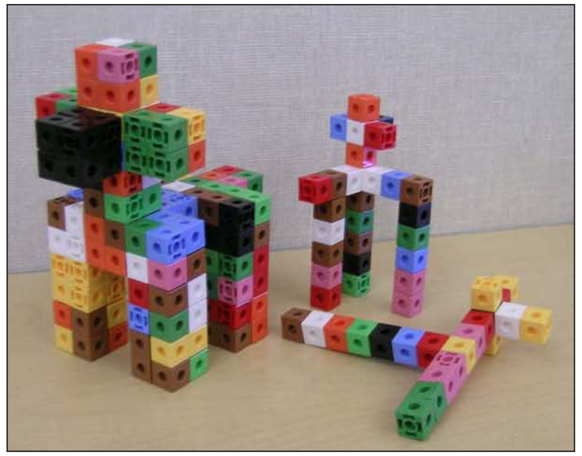

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Fig. 5 The completed dog looms large over the puppy.

stormed terms, an activity that affords an excellent context for the necessity of accurate and uniform vocabulary in mathematics.

After students work through the meanings of their commonly generated terminology via class discussion, they usually determine that *volume* is the best way for describing the "size" of a dog, whereas *surface area* in some way relates to the "skin" of the dog and its ability to cool. Although students usually determine that surface area is a key concept pertaining to how dogs cool off, they often have difficulty justifying why surface area is relevant to cooling. To better provide concrete intellectual necessity for why surface area is important in the lesson, the teacher gives a brief demonstration so that students can link their prior, more intuitive, knowledge of cooling with the concept of surface area.

A "COOL" DEMONSTRATION

The surface area cooling demonstration begins with the teacher presenting two equal-volume Styrofoam™ cups of water that had previously been boiling at approximately 100°C. After showing that the thermometer readings are identical for the two cups, the teacher pours one cup into a shallow plastic rectangular pan (approximately 20×30 cm) and pours the other cup into an identical cup. Both containers are at room temperature. While the water is cooling in the separate containers, the teacher asks the class to discuss the setup of the experiment to ensure that everyone will grasp the experiment's outcome. After students understand that equal volumes of water at the same temperatures are poured into containers with different surface area, the teacher measures the temperatures of the pan and cup of water, showing that the water in the pan is significantly cooler than the water in the cup (45°C cooler within 5 minutes).

The following dialogue excerpt captures students encountering an important moment of cognitive disequilibrium as they discuss the surface area cooling demonstration in the context of the Cooling Dog problem:

Student 1: The pan of water cooled faster because the water is thinner, like my Chihuahua is skinny more than a big dog, so it gets cold easier.
Student 2: The cup of water has the same amount as the pan. The pan is more spread out, so it hits the air more. Your Chihuahua is not as big as a big dog, so it's not the same!
Student 3: Yeah, the area is bigger on the pan like the area is bigger for big dogs.
Student 1: Then the big dog is cold more … but they get hot more … don't they?
Student 2: Maybe they have more hair than little dogs, so they get hot more.
Student 4: We have a really big Great Dane that's always panting, but he has short fur.
Student 1: Yeah … the big dog has more volume than the little dog, so it's different than the water in the pan, 'cause they had the same amount. Now I see what you meant!

As the groups settle on the notion that surface area is a mathematical quantity related to both the models' sizes and how fast objects cool, they also tend to become aware that large dogs, with more surface area than small dogs, will cool off faster. So why do large dogs appear to be hot more than small dogs, as students originally observed from the photographs? Having students suggest this puzzling question is an intentional component of the lesson design, motivating them to see the necessity of the ratio concept for understanding the change of surface area in relation to volume.

COUNTING, CALCULATING, AND RESOLVING

Now that students have constructed their models, they can focus directly on computing the surface areas and volumes of the small and large dogs. Although volume is usually a simple matter of counting blocks, many students tend to neglect counting the exposed surface areas on the insides of the legs, shoulders, bottoms of feet, and ears. However, these problems are left to students to work out and discover in groups; the teacher gives feedback simply in terms of right or wrong surface area and volume computations.

Some groups attempt to calculate the surface area and volume of the large dog through a fairly tedious process of counting faces and blocks, whereas other groups often discover that "a block" of the small dog is a $2 \times 2 \times 2$ cube of blocks for the large-dog model. When one group of students was asked to discuss the quick speed of their calculations, they explained that each block of the small dog is 8 blocks for the large dog, so the new volume is $8 \times 22u^3 = 176u^3$ (u = units). Similarly, because the surface of just the nose of the large dog consists of 4 small-dog $1u^2$ nose surfaces, then the surface area of the entire large dog is $4 \times 90u^2 = 360u^2$.

Common student observations based on the computational results include the following:

- The large dog has more surface area than the small dog and hence more surface area to cool itself (in agreement with the surface area cooling demonstration).
- The large dog has a greater volume to cool off than the small dog.

The second observation is important to students' unlocking the proportional reasoning clues needed to resolve the Cooling Dog problem, as exemplified in the following dialogue:

Student 1: The big dog has more area than the little dog to cool off, but the big dog has a lot more that needs to cool off!
Student 2: Yeah, if they were the same size, the big dog would be cold, but the big dog is a lot bigger! They got a lot more they got to cool … and so not as much as the little dog does.

The preceding dialogue suggests that, from a constructivist perspective, students are making connections among surface area, volume, and cooling. These connections may eventually help restore coherence to thinking that tends to view the models strictly with surface area as *distinct* from volume rather than as surface area that changes in relation to changing volume. Equipped with such insights, students are ready to use proportional reasoning to address questions surrounding the Cooling Dog problem mathematically.

SURFACE-AREA-TO-VOLUME RATIO

Having now reached a point in the lesson of achieving a multifaceted motivation for the ratio concept, the teacher formally introduces the surface-area-to-volume ratio to the students as a useful ratio needed to understand the problems they have encountered. Working with the ratio in groups, students calculate that the small dog has an average of

$$\frac{SA_{\text{small}}}{V_{\text{small}}} = \frac{90u^2}{22u^3} = \frac{4.09u^2}{1u^3},$$

whereas the doubled dog has a ratio of

$$\frac{SA_{\text{large}}}{V_{\text{large}}} = \frac{360u^2}{176u^3} = \frac{2.05u^2}{1u^3}.$$

To resolve their previous questions, students can now reflect on the large-dog ratio in several different ways.

Students can see that

$$\left(\frac{SA}{V}\right)_{\text{large}} = \frac{360u^2}{176u^3} = \frac{4 \bullet 90u^2}{8 \bullet 22u^3}$$

$$= \frac{1}{2} \bullet \left(\frac{SA}{V}\right)_{\text{small}} = \frac{\frac{1}{2} \bullet SA_{\text{small}}}{V_{\text{small}}},$$

which shows that the situation for a large dog is comparable to that of a small dog having half as much surface area. The equation

$$\left(\frac{SA}{V}\right)_{\text{large}} = \frac{SA_{\text{small}}}{2 \bullet V_{\text{small}}}$$

shows that the large dog's cooling capacity is like that of two small dogs having only the skin of one small dog to cool. The teacher can also guide students to the mathematical explanation of the role of panting in cooling. For example, a dog's panting involves moving the flat tongue *outside* the body, where it can function as a radiator, liberating heat by increasing the dog's surface area without increasing its volume.

From the large ears of the elephant to the structure of the lung (holes simultaneously increase surface area and decrease volume, greatly increasing the *S*/*V* ratio), the surface-area-to-volume ratio is a rich concept that may help interpret many adaptations in biological structures. The surface-area-to-volume ratio not only influences the development of biological structures but also can affect metabolic rates. For example, large animals tend to have slower metabolic rates; they have lower surface-area-to-volume ratios than other small (homeothermic) mammals, such as mice, and as a result they lose less heat per gram (Wells 2007).

CONCLUSION: GENERALIZING FROM DOGS TO GIANTS

Students learn to generalize doubling by proportional reasoning and reflection on the fact that a third doubling of the puppy model would lead to the following result:

$$\left(\frac{SA}{V}\right)_{\text{large}} = \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{SA_{\text{small}}}{V_{\text{small}}}$$

Using rulers, students then calculate that for the 300-foot giant in **figure 1**, an assumed 5-foot-tall "small man" near the giant's ankle is doubled approximately 6 times, giving

$$\frac{SA_{300\text{ ft.}}}{V_{300\text{ ft.}}} \approx \left(\frac{1}{2}\right)^6 \bullet \frac{S_{5\text{ ft.}}}{V_{5\text{ ft.}}},$$

or 1/64 of the original surface-area-to-volume ratio. Compared with the small man, the giant has less than 1/64 of the skin-cooling capacity, or, alternatively, 64 of the small men have only the surface area or "skin" of one small man with which to cool off. Without structural or metabolic adaptations, a giant such as this would most certainly be more comfortable in extremely cold climates—if he could even exist at all without overheating.

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LANCE BURGER, lburger@csufresno.edu, is an assistant professor of mathematics at California State University, Fresno. He is interested in the psychology of mathematical thinking and applying mathematics education research to the classroom and professional development.